**Time limit:** 45 minutes.
**No calculators**
**Instructions** The following questions are meant to be solved by a team of up to four people and require full, well-formulated proofs as answers. Partial credit will be given to solutions that do not completely answer a problem or those with errors. If you do not finish a question you may still the results of that question to solve later parts of the exam.

# Introduction

This round will introduce a particular extension of the rational numbers that satisfies certain properties detailed below. By an extension of the rational numbers we mean a set that contains the rational numbers. This is denoted by the following: if A contains the rationals we write $\mathbb{Q} \subset A$ or $A \supset \mathbb{Q}$. Some familiar examples of extensions of the rationals are the complex numbers and the real numbers, which will be denoted by $\mathbb{C}$ and $\mathbb{R}$ respectively. The rationals and integers will be denoted by $\mathbb{Q}$ and $\mathbb{Z}$.

Before jumping right into the test, we begin with some definitions:

**Definition 1** *A set $A \supset \mathbb{Q}$ is called a field if it satisfies the following properties:*
*i) If $x,y \in A$, that is, $x$ and $y$ are in $A$, then so are $xy$ and $x+y$*
*ii) Both 0 and 1 are elements of $A$*
*iii) If $x \in A$ then so is $-x$. If, in addition, $x \neq 0$ then $\frac{1}{x}$ is in $A$ as well.*

**Example 1** *$\mathbb{Q}$ is itself a field*

**Example 2** *The set of irrational numbers, $\mathbb{I}$ is not a field since $\sqrt{2} * \sqrt{2} = 2$ which is not irrational so $\mathbb{I}$ fails to satisfy property i of the definition of a field.*

**Definition 2** *If $A \supset \mathbb{Q}$ is a field, then we call $A$ an ordered field if $A$ satisfies the following properties:*
*i) For any $x, y \in A$ if $x \neq y$ then either $x < y$ or $y < x$*
*ii) If $x, y \geq 0$ then $xy \geq 0$*
*iii) If $x \leq y$ then for any $z \in A$, $x + z \leq y + z$*

**Definition 3** *Let $A \supset \mathbb{Q}$ be an ordered field. The Least Upper Bound Property states the following: If $X \subset A$ has an upper bound (there exists $M \in A$ such that for all $a \in X$, $M > a$) then there is an element $b \in A$, called the least upper bound, that satisfies the following properties:*
*i) $b$ is an upper bound of $X$*
*ii) If $a \in A$ and $a < b$ then there is $c \in X$ such that $a < c \leq b$*

**Example 3** *The least upper bound of the set $\{x \in \mathbb{Q} : x < 2\}$ is 2.*

# Questions

Let $A \supset \mathbb{Q}$ be an ordered field satisfying the Least Upper Bound Property.

1. Show that $A \neq \mathbb{C}$ (4 points)

   For the remainder of the test you may assume that $A \subset \mathbb{R}$

2. Show that $A \neq \mathbb{Q}$ (4 points)

3. Let $a, b \in A$ such that $0 < a < b$. Show, using the least upper bound property, that there is an integer n such that $na > b$. (4 points)

4. Let $a, b \in A$ such that $0 < a < b$. Show that there exists infinitely many $c \in \mathbb{Q}$ such that $a < c < b$ (4 points)

   **Definition 4** *The Greatest Lower Bound Property states the following:*
   *If $X \subset A$ has a lower bound (there exists $M \in A$ such that for all $a \in X$, $M < a$) then there is an element $b \in A$, called the greatest lower bound, that satisfies the following properties:*
   *i) $b$ is a lower bound of $X$*
   *ii) if $a \in A$ and $a > b$ then there is $c \in X$ such that $a > c \geq b$*

5. Show that A satisfies the Greatest Lower Bound Property (6 points)

   **Definition 5** *For any $a, b \in A$ define the interval $I_{ab} = \{x \in A : a \leq x \leq b\}$. We also define the length of an interval $Length(I_{ab})$ to be $b - a$.*

6. Let $I_{a_1b_1} \supset I_{a_2b_2} \supset I_{a_3b_3} \supset \ldots$ be an infinite collection of intervals in A. Show that $\bigcap_{n=1}^{\infty} I_{a_nb_n} = \{x \in A : x \in I_{a_nb_n}, \text{ for all } n \in \mathbb{Z}\}$ is not empty. This is called the Nested Interval Theorem. (12 points)

7. Let $I_{a_1b_1}, I_{a_2b_2}, I_{a_3b_3}\ldots$ be as in question 6. Assume also that $\lim_{n\to\infty} Length(I_{a_nb_n}) = 0$. That is, for any $\epsilon > 0$ there is an N such that for all $n \geq N$ $Length(I_{a_nb_n}) < \epsilon$. Show that $\bigcap_{n=1}^{\infty} I_{a_nb_n} = x$ for some $x \in A$. (4 points)

8. Let $B \supset \mathbb{Q}$ be an ordered field that satisfies the Nested Interval Theorem. Show that B also satisfies the Least Upper Bound Property. (12 points)

   An ordered field satisfying either the Nested Interval Theorem or the Least Upper Bound Property is called complete.